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NUCLEARITY OF REDUCED AMALGAMATED FREE PRODUCT C*-ALGEBRAS

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ABSTRACT. We show that the reduced free product between two nuclear C*-algebras is nuclear provided that at least one of the states is pure. We also show that some reduced amalgamated free product C*-algebras are nuclear. These generalize an unpublished work of Dykema and Smith and verify Dykema's conjecture. Our proof is modeled after Dykema's work on exactness of reduced amalgamated free product C*-algebras.

1. INTRODUCTION

The reduced amalgamated free product of C*-algebras was introduced by Voiculescu [Vo] [VDN]. Dykema [Dy] proved that the reduced amalgamated free product of exact C*-algebras is exact. We will follow his paper [Dy] the notation. Dykema and Smith [DS] proved, among other things, that if ω is a pure state on the full matrix algebra M_n and ϕ is a state on a nuclear C*-algebra A , then the reduced free product $(M_n, \omega) * (A, \phi)$ is nuclear. We generalize their result and verify Dykema's conjecture that the reduced free product between two nuclear C*-algebras is nuclear provided that at least one of the states is pure.

Theorem 1.1. *Let B be a unital C*-algebra and let A_i ($i = 1, 2$) be a unital nuclear C*-algebra containing B as a unital C*-algebra and having a conditional expectation ϕ_i from A_i onto B , whose GNS representation is faithful. Let $(A, \phi) = (A_1, \phi_1) * (A_2, \phi_2)$ be the reduced amalgamated free product of C*-algebras. Suppose either (i) ϕ_1 is a pure state (and $B = \mathbb{C}1$) or (ii) $\mathbb{K}(E_1) \subset A_1$ in its GNS representation. Then, A is nuclear.*

We remark that the same assertion holds if one replaces nuclearity with one of the following, the LLP, the WEP and exactness. In particular, we recover Dykema's theorem [Dy].

We recall that a C*-algebra A is nuclear if $A \otimes C = A \otimes_{\max} C$ for any C*-algebra C . It is well known that the class of nuclear C*-algebras is closed under (i) passing to a C*-subalgebra that is a range of a conditional expectation, (ii) passing to a quotient and (iii) taking an extension.

We recall the definition of reduced amalgamated free product. See [Dy] [Vo] for the detail. Let B be a unital C*-algebra and let A_i ($i = 1, 2$) be a unital nuclear C*-algebra containing B as a unital C*-algebra and having a conditional expectation ϕ_i from A_i onto B such that for any nonzero $a \in A_i$, there is $x \in A_i$ with $\phi_i(x^* a^* a x) \neq 0$. Let $\langle \cdot, \cdot \rangle_{E_i}$ be the B -valued inner product on A_i , given by $\langle a, b \rangle_{E_i} = \phi_i(a^* b)$, and let E_i be the Hilbert B -module obtained from A_i by separation and completion. We denote by \widehat{a} the element in E_i arising from a in A_i . Then A_i is faithfully represented on E_i by $\widehat{ab} = \widehat{a}\widehat{b}$. Let $\xi_i = \widehat{1_{A_i}} \in E_i$ be the distinguished element and let E_i° be the complementing B -submodule of $\xi_i B$ in E_i , i.e., $E_i = \xi_i B \oplus E_i^\circ$. Then the reduced amalgamated free product C*-algebra $(A, \phi) = (A_1, \phi_1) * (A_2, \phi_2)$ is a C*-subalgebra of $\mathbb{B}(E)$ generated by copies of A_1 and A_2 , where

$$E = \xi B \oplus \bigoplus_{\substack{n \in \mathbb{N}, i_1, \dots, i_n \in \{1, 2\}, \\ i_1 \neq i_2, \dots, i_{n-1} \neq i_n}} E_{i_1}^\circ \otimes_B \cdots \otimes_B E_{i_n}^\circ.$$

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Here ξB is the C^* -algebra B regarded as a Hilbert B -module with the distinguished element $\xi = 1_B$. For $a \in A_i^\circ = A_i \cap \ker \phi_i$ and for $\zeta_j \in E_{i_j}^\circ$ ($j = 1, \dots, n, n \geq 2$) with $i_1 \neq \dots \neq i_n$, we have

$$a(\zeta_1 \otimes \dots \otimes \zeta_n) = \begin{cases} (a\zeta_1 - \xi_i \langle \xi_i, a\zeta_1 \rangle) \otimes \zeta_2 \otimes \dots \otimes \zeta_n & \text{if } i_1 = i \\ \quad + \langle \xi_i, a\zeta_1 \rangle \zeta_2 \otimes \dots \otimes \zeta_n & \\ \widehat{a} \otimes \zeta_1 \otimes \dots \otimes \zeta_n & \text{if } i_1 \neq i. \end{cases}$$

The reduced amalgamated free product C^* -algebra A is the closed linear span of B and the elements of the form $a_1 \dots a_n$ where $n \in \mathbb{N}$, $a_j \in A_{i_j}^\circ$ with $i_1 \neq \dots \neq i_n$.

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2. PROOF OF THEOREM 1.1

The case (i) immediately follows from Choda's proposition [Ch] (see also [BD]), Kishimoto and Sakai's lemma [KS] and the result of Dykema and Smith [DS] (or the case (ii) of our theorem). We now concentrate our attention on the case (ii) and assume that $\mathbb{K}(E_1) \subset A_1$. We may assume that A_1 and A_2 are separable thanks to Blanchard and Dykema's theorem [BD].

Following [Dy], we will construct, in this section, a sequence of ucp maps which approximates the identity map on A . Let $p_1 \in \mathbb{B}(E_1)$ be the projection from E_1 onto $\xi_1 B$. Then $p_1 \in A_1$ by the assumption. For $\zeta_j \in E_{i_j}^\circ$ ($j = 1, \dots, n, n \geq 2$) with $i_1 \neq \dots \neq i_n$, we have

$$\begin{aligned} \text{if } a \in p_1 A_1 (1 - p_1), \text{ then} \quad & a(\zeta_1 \otimes \dots \otimes \zeta_n) = \begin{cases} \langle \xi_i, a\zeta_1 \rangle \zeta_2 \otimes \dots \otimes \zeta_n & \text{if } i_1 = 1 \\ 0 & \text{if } i_1 = 2, \end{cases} \\ \text{if } a \in (1 - p_1) A_1 (1 - p_1), \text{ then} \quad & a(\zeta_1 \otimes \dots \otimes \zeta_n) = \begin{cases} a\zeta_1 \otimes \zeta_2 \otimes \dots \otimes \zeta_n & \text{if } i_1 = 1 \\ 0 & \text{if } i_1 = 2, \end{cases} \\ \text{if } a \in (1 - p_1) A_1 p_1, \text{ then} \quad & a(\zeta_1 \otimes \dots \otimes \zeta_n) = \begin{cases} 0 & \text{if } i_1 = 1 \\ \widehat{a} \otimes \zeta_1 \otimes \dots \otimes \zeta_n & \text{if } i_1 = 2. \end{cases} \end{aligned}$$

We let, for $m, n \geq 0$,

$$X_{(m,0,n)} = \{b_0 a_1 b_1 \dots a_{m+n} b_{m+n} : b_0, b_{m+n} \in \mathbb{C}1 \cup A_2^\circ, b_1, \dots, b_{m+n-1} \in A_2^\circ, \\ a_1 \dots, a_m \in (1 - p_1) A_1 p_1, a_{m+1}, \dots, a_{m+n} \in p_1 A_1 (1 - p_1)\}$$

and

$$X_{(m,1,n)} = \{b_0 a_1 b_1 \dots a_{m+n+1} b_{m+n+1} : b_0, b_{m+n+1} \in \mathbb{C}1 \cup A_2^\circ, b_1, \dots, b_{m+n} \in A_2^\circ, \\ a_1 \dots, a_m \in (1 - p_1) A_1 p_1, a_{m+1} \in (1 - p_1) A_1 (1 - p_1), \\ a_{m+2}, \dots, a_{m+n+1} \in p_1 A_1 (1 - p_1)\}$$

be the subsets of A .

Lemma 2.1. *We have*

$$(1 - p_1)b(1 - p_1) = \phi_2(b)(1 - p_1)$$

for $b \in A_2$. In particular,

$$A = \overline{\text{span}} \bigcup_{m,n \geq 0} (X_{(m,0,n)} \cup X_{(m,1,n)}).$$

Proof. Straightforward. 

We define Hilbert B -submodules F_l of E ($l = 0, 1, \dots$) by $F_0 = \xi B \oplus E_2^\circ$ and

$$F_l = \overbrace{E_1^\circ \otimes_B \cdots \otimes_B E_1^\circ}^{2l-1} \oplus \overbrace{E_1^\circ \otimes_B \cdots \otimes_B E_2^\circ}^{2l} \oplus \overbrace{E_2^\circ \otimes_B \cdots \otimes_B E_1^\circ}^{2l} \oplus \overbrace{E_2^\circ \otimes_B \cdots \otimes_B E_2^\circ}^{2l+1}$$

for $l = 1, 2, \dots$ and put $F_{(\rightarrow k)} = \bigoplus_{l=0}^k F_l$. We observe that $E = \bigoplus_{l=0}^\infty F_l$. Let Q_l (resp. $Q_{(\rightarrow k)}$) be the projection from E onto F_l (resp. from E onto $F_{(\rightarrow k)}$).

Lemma 2.2. *We have the following.*

$$\begin{aligned} bQ_l &= Q_lb & \text{if } b \in A_2, & & aQ_l &= Q_{l-1}a & \text{if } a \in p_1 A_1 (1 - p_1) \\ aQ_l &= Q_la & \text{if } a \in (1 - p_1) A_1 (1 - p_1), & & aQ_l &= Q_{l+1}a & \text{if } a \in (1 - p_1) A_1 p_1 \end{aligned}$$

Proof. Straightforward. \square

We define the isometry $V_k: E \rightarrow \ell_2 \otimes_{\mathbb{C}} F_{(\rightarrow k)} \otimes_B E$ ($k = 0, 1, \dots$) by

$$\begin{aligned} \eta &\longmapsto \delta_{k-l} \ddot{\otimes} \eta \ddot{\otimes} \xi & \text{if } \eta \in F_l \text{ with } l \leq k \\ V_k: \zeta_1 \otimes \cdots \otimes \zeta_m &\longmapsto \delta_0 \ddot{\otimes} (\zeta_1 \otimes \cdots \otimes \zeta_{2k}) \ddot{\otimes} (\zeta_{2k+1} \otimes \cdots \otimes \zeta_m) & \text{if } \zeta_1 \in E_1^\circ \text{ and } m > 2k \\ \zeta_1 \otimes \cdots \otimes \zeta_m &\longmapsto \delta_0 \ddot{\otimes} (\zeta_1 \otimes \cdots \otimes \zeta_{2k+1}) \ddot{\otimes} (\zeta_{2k+2} \otimes \cdots \otimes \zeta_m) & \text{if } \zeta_1 \in E_2^\circ \text{ and } m > 2k + 1. \end{aligned}$$

The symbol $\ddot{\otimes}$ is used in order to distinguish various tensor products.

Lemma 2.3. *The isometries $\{V_k\}_{k=0}^\infty$ have mutually orthogonal ranges and satisfy the following.*

$$\begin{aligned} (1 \otimes b \otimes 1)V_k &= V_k b & \text{if } b \in A_2, & & (1 \otimes a \otimes 1)V_k &= V_{k-1}a & \text{if } a \in p_1 A_1 (1 - p_1) \\ (1 \otimes a \otimes 1)V_k &= V_k a & \text{if } a \in (1 - p_1) A_1 (1 - p_1), & & (1 \otimes a \otimes 1)V_k &= V_{k+1}a & \text{if } a \in (1 - p_1) A_1 p_1 \end{aligned}$$

Proof. Straightforward. \square

Letting

$$V_{(\rightarrow N)} = \frac{1}{\sqrt{N+1}} \sum_{k=0}^N V_k$$

be the isometry from E into $\ell_2 \otimes_{\mathbb{C}} F_{(\rightarrow N)} \otimes_B E$, we consider the compression

$$\Phi_N: \mathbb{B}(E) \rightarrow \mathbb{B}(F_{(\rightarrow N)})$$

and the ucp map $\Psi_N: \mathbb{B}(F_{(\rightarrow N)}) \rightarrow \mathbb{B}(E)$ defined by

$$\Psi_N(x) = V_{(\rightarrow N)}^* (1 \otimes x \otimes 1) V_{(\rightarrow N)}$$

and we put $\Theta_N = \Psi_N \circ \Phi_N$.

Lemma 2.4. *The ucp map Θ_N maps A into A and*

$$\lim_{N \rightarrow \infty} \|x - \Theta_N(x)\| = 0$$

for every $x \in A$. Indeed, if $x \in X_{(m,0,n)} \cup X_{(m,1,n)}$ then

$$\Theta_N(x) = \max\{0, \min\{1 - m/(N+1), 1 - n/(N+1)\}\}x.$$

Proof. This follows from Lemma 2.3. \square

We define the ‘diagonal’ C^* -subalgebra D in A as

$$D = \overline{\text{span}} \bigcup_{l \geq 0} (X_{(l,0,l)} \cup X_{(l,1,l)}).$$

This is indeed a C^* -algebra as shown below.

Lemma 2.5. *If $x \in X_{(l,c,l)}$ and $y \in X_{(l',c',l')}$ with $l \geq l' \geq 0$ and $c, c' \in \{0, 1\}$, then*

$$xy \in \text{span}(X_{(l,c,l)} \cup X_{(l,1,l)}).$$

Proof. This follows from Lemma 2.1 and the equation

$$(1 - p_1)bab'(1 - p_1) = \phi_2(b\phi_1(a)b')(1 - p_1)$$

for $a \in A_1$ and $b, b' \in A_2^\circ$. \mathfrak{M}

We define the ucp map $\Delta: A \rightarrow \mathbb{B}(E)$ by

$$\Delta(x) = \text{strict-} \sum_{l=0}^{\infty} Q_l x Q_l.$$

Lemma 2.6. *The ucp map Δ is a conditional expectation from A onto D and*

$$D = \{x \in A : xQ_l = Q_l x \text{ for all } l = 0, 1, \dots\}.$$

Proof. This follows from Lemmas 2.1 and 2.2. \mathfrak{M}

Lemma 2.7. *The C^* -algebra D is nuclear.*

We postpone the proof to Section 3 and now prove Theorem 1.1.

Proof of Theorem 1.1. We may assume that there is $a \in A_1^\circ$ (resp. $b \in A_2^\circ$) such that $\phi_1(a^*a) = 1$ (resp. $\phi_2(b^*b) = 1$). Indeed, we just replace $(B \subset A_1, \phi_1)$ with $(B \otimes \mathbb{C}I \subset A_1 \otimes \mathbb{M}_2, \phi_1 \otimes \psi)$, where $\psi(x) = x_{11}I$ for $x = [x_{ij}] \in \mathbb{M}_2$, and let $a = 1 \otimes e_{21}$. We observe that the conditional expectation $\text{id} \otimes \psi$ from $A_1 \otimes \mathbb{M}_2$ onto A_1 is compatible, in the sense of [BD], with the conditional expectations ϕ_1 and $\phi_1 \otimes \psi$. The same for A_2 . Since $A = (A_1, \phi_1) * (A_2, \phi_2)$ is a C^* -subalgebra of $\tilde{A} = (A_1 \otimes \mathbb{M}_2, \phi_1 \otimes \psi) * (A_2 \otimes \mathbb{M}_2, \phi_2 \otimes \psi)$ and is a range of the conditional expectation $(\phi_1 \otimes \psi) * (\phi_2 \otimes \psi)$, the nuclearity of A follows from that of \tilde{A} . We have used here Theorems 1.3 and 2.2 in [BD].

Let a and b be as above. Then, $w = bab_1 + ab(1 - p_1)$ is an isometry in A with $wQ_l = Q_{l+1}w$ for $l = 0, 1, \dots$ and hence we have $wxw^* \in D$ for all $x \in D$. Therefore, A is $*$ -isomorphic to the crossed product D by the $*$ -endomorphism $\text{Ad } w$ and a fortiori is nuclear. \mathfrak{M}

3. PROOF OF LEMMA 2.7

Let $D_{(\rightarrow l)} = Q_{(\rightarrow l)}D$ be the C^* -subalgebra of $\mathbb{B}(F_{(\rightarrow l)})$. Since the ucp map Θ_N appearing in Lemma 2.4 maps D into D and factors through $D_{(\rightarrow N)}$, Lemma 2.7 follows from the nuclearity of $D_{(\rightarrow l)}$ ($l = 0, 1, \dots$). Following [Dy], we will prove this by induction.

Lemma 3.1. *If $x \in (\bigcup_{m \geq l+1} X_{(m,0,m)}) \cup (\bigcup_{m \geq l} X_{(m,1,m)})$, then $Q_{(\rightarrow l)}x = 0$. In particular,*

$$D_{(\rightarrow l)} = Q_{(\rightarrow l)} \overline{\text{span}}\left(\left(\bigcup_{m=0}^l X_{(m,0,m)}\right) \cup \left(\bigcup_{m=0}^{l-1} X_{(m,1,m)}\right)\right).$$

Proof. Straightforward. \mathfrak{M}

Let

$$F_l^\circ = \overbrace{E_1^\circ \otimes_B \cdots \otimes_B E_2^\circ}^{2l} \oplus \overbrace{E_2^\circ \otimes_B \cdots \otimes_B E_2^\circ}^{2l+1}$$

for $l = 0, 1, \dots$ and let Q_l^\bullet be the projection from E onto F_l^\bullet . We define the unitary $W_l: F_l \rightarrow F_{l-1}^\bullet \otimes_B E_1^\circ \otimes_B E_2$ ($l = 1, 2, \dots$) by

$$W_l: \begin{array}{ll} \zeta_1 \otimes \dots \otimes \zeta_{2l-1} & (\zeta_1 \otimes \dots \otimes \zeta_{2l-2}) \otimes \zeta_{2l-1} \otimes \xi_2 \text{ if } \zeta_1 \in E_1^\circ \\ \zeta_1 \otimes \dots \otimes \zeta_{2l} & (\zeta_1 \otimes \dots \otimes \zeta_{2l-2}) \otimes \zeta_{2l-1} \otimes \zeta_{2l} \text{ if } \zeta_1 \in E_1^\circ \\ \zeta_1 \otimes \dots \otimes \zeta_{2l} & (\zeta_1 \otimes \dots \otimes \zeta_{2l-1}) \otimes \zeta_{2l} \otimes \xi_2 \text{ if } \zeta_1 \in E_2^\circ \\ \zeta_1 \otimes \dots \otimes \zeta_{2l+1} & (\zeta_1 \otimes \dots \otimes \zeta_{2l-1}) \otimes \zeta_{2l} \otimes \zeta_{2l+1} \text{ if } \zeta_1 \in E_2^\circ \end{array} \mapsto$$

and the ucp map $\sigma_l: D_{(\rightarrow(l-1))} \rightarrow \mathbb{B}(F_{(\rightarrow(l-1))} \oplus F_l) = \mathbb{B}(F_{(\rightarrow l)})$ by

$$\sigma_l(x) = x \oplus W_l^*(Q_{l-1}^\bullet x Q_{l-1}^\bullet \otimes 1 \otimes 1) W_l$$

for $x \in D_{(\rightarrow(l-1))} \subset \mathbb{B}(F_{(\rightarrow(l-1))})$. We define C^* -subalgebras I_l^\bullet and $D_{(\rightarrow l)}^\bullet$ of $D_{(\rightarrow l)}$ by

$$I_l = Q_{(\rightarrow l)} \overline{\text{span}} X_{(l-1,1,l-1)}$$

and

$$D_{(\rightarrow l)}^\bullet = Q_{(\rightarrow l)} \overline{\text{span}}((\bigcup_{m=0}^{l-1} X_{(m,0,m)}) \cup (\bigcup_{m=0}^{l-1} X_{(m,1,m)})).$$

It follows from Lemma 2.5 that these are indeed C^* -algebras and that I_l is an ideal of $D_{(\rightarrow l)}^\bullet$.

Lemma 3.2. *The ucp map σ_l maps $D_{(\rightarrow(l-1))}$ into $D_{(\rightarrow l)}^\bullet$. Indeed, if $x \in (\bigcup_{m=0}^{l-1} X_{(m,0,m)}) \cup (\bigcup_{m=0}^{l-2} X_{(m,1,m)})$, then*

$$\sigma_l(Q_{(\rightarrow(l-1))} x) = Q_{(\rightarrow l)} x.$$

Proof. Straightforward. \square

Lemma 3.3. *If $D_{(\rightarrow(l-1))}$ is nuclear, then $D_{(\rightarrow l)}^\bullet$ is nuclear.*

Proof. Let π_l be the quotient map from $D_{(\rightarrow l)}^\bullet$ onto $D_{(\rightarrow l)}^\bullet / I_l$ and let $\rho_l = \pi_l \circ \sigma_l$. We claim that ρ is a $*$ -homomorphism from $D_{(\rightarrow(l-1))}$ onto $D_{(\rightarrow l)}^\bullet / I_l$. In view of Lemma 3.1, it suffices to show the multiplicativity of ρ_l on $Q_{(\rightarrow(l-1))}((\bigcup_{m=0}^{l-1} X_{(m,0,m)}) \cup (\bigcup_{m=0}^{l-2} X_{(m,1,m)}))$, but this follows from Lemmas 2.5 and 3.2. In particular, $D_{(\rightarrow l)}^\bullet / I_l$ is nuclear by the assumption.

We next show that I_l is nuclear. Then the nuclearity of $D_{(\rightarrow l)}^\bullet$ follows. As $I_l = Q_l I_l$ by Lemma 3.1, I_l is $*$ -isomorphic to a C^* -subalgebra of $\mathbb{B}(F_{l-1}^\bullet \otimes_B E_1^\circ \otimes_B E_2)$ via the unitary W_l . We claim that this C^* -subalgebra is $(\mathbb{K}(F_{l-1}^\bullet) \rtimes (1 - p_1) A_1 (1 - p_1)) \otimes \mathbb{C} 1$ and thus I_l is nuclear. See [Dy] for the definition and the property of the operation \rtimes . Indeed, if $x \in X_{(l-1,1,l-1)}$ is, e.g., of the form

$$x = b_0 a_1 b_1 \dots b_{2l-2} a_{2l-1} b_{2l-1}$$

with $a_1, \dots, a_{l-1} \in (1 - p_1) A_1 p_1$, $a_l \in (1 - p_1) A_1 (1 - p_1)$, $a_{l+1}, \dots, a_{2l-1} \in p_1 A_1 (1 - p_1)$ and $b_0, \dots, b_{2l-1} \in A_2^\circ$, then

$$W_l x W_l^* = (\theta_{\widehat{b_0} \otimes \widehat{a_1} \otimes \dots \otimes \widehat{b_{l-1}}} a_l \theta_{\widehat{b_{2l-1}^*} \otimes \widehat{a_{2l-1}^*} \otimes \dots \otimes \widehat{b_1^*}}) \otimes 1.$$

The other cases are similar and this completes the proof. \square

We define the ideal J_l of $D_{(\rightarrow l)}$ by

$$J_l = Q_{(\rightarrow l)} \overline{\text{span}} X_{(l,0,l)}.$$

It follows from Lemmas 2.5 and 3.1 that this is indeed an ideal.

Lemma 3.4. *If $D_{(\rightarrow(l-1))}$ is nuclear, then $D_{(\rightarrow l)}$ is nuclear.*

Proof. Assume that $D_{(\rightarrow, l-1)}$ is nuclear. Then the nuclearity of $D_{(\rightarrow, l)}$ follows from that of J_l thanks to Lemma 3.3 and the fact $D_{(\rightarrow, l)} = D_{(\rightarrow, l)}^* + J_l$. As $J_l = Q_l J_l$ by Lemma 3.1, J_l is $*$ -isomorphic to a C^* -subalgebra of $\mathbb{B}(F_{l-1}^* \otimes_B E_1^\circ \otimes_B E_2)$ via the unitary W_l . We claim that this C^* -subalgebra is $\mathbb{K}(F_{l-1}^* \otimes_B E_1^\circ) \rtimes A_2$ and thus J_l is nuclear. Indeed, if $x \in X_{(l, 0, l)}$ is, e.g., of the form

$$x = b_0 a_1 b_1 \cdots b_{2l-1} a_{2l} b_{2l}$$

with $a_1, \dots, a_l \in (1 - p_1)A_1 p_1$, $a_{l+1}, \dots, a_{2l} \in p_1 A_1 (1 - p_1)$ and $b_0, \dots, b_{2l} \in A_2^\circ$, then

$$W_l x W_l^* = \theta_{\widehat{b_0} \otimes \widehat{a_1} \otimes \cdots \otimes \widehat{a_l}} b_l \theta_{\widehat{b_{2l}} \otimes \widehat{a_{2l}} \otimes \cdots \otimes \widehat{a_{l+1}}}.$$

The other cases are similar and this completes the proof. \square

Proof of Lemma 2.7. As we mentioned, it suffices to show that $D_{(\rightarrow, l)}$ is nuclear for $l = 0, 1, \dots$, but this follows from Lemma 3.4 and the fact that $D_{(\rightarrow, 0)} = A_2$ is nuclear. \square

APPENDIX A. AN ELEMENTARY PROOF OF THE BDS THEOREM ON TOPOLOGICAL ENTROPY OF FREE PRODUCT AUTOMORPHISMS

We give an elementary proof of the following theorem of Brown, Dykema and Shlyakhtenko [BDS]. What they actually proved is more general, namely the same assertion for amalgamated free products with finite dimensional amalgams. Consult [Br] for the definition of the topological entropy.

Theorem A.1. *Let A_i ($i = 1, 2$) be a unital exact C^* -algebra with a $*$ -automorphism α_i and let $\alpha = \alpha_1 * \alpha_2$ be the free product $*$ -automorphism (defined in [Ch]) on the reduced free product C^* -algebra $A = A_1 * A_2$. Then, we have*

$$\text{ht}(\alpha) = \max\{\text{ht}(\alpha_1), \text{ht}(\alpha_2)\},$$

where ht is the Brown-Voiculescu topological entropy.

The Kadison-Schwarz inequality implies that a ucp map ϕ satisfies

$$\|\phi(x^* y) - \phi(x)^* \phi(y)\| \leq \|\phi(x^* x) - \phi(x)^* \phi(x)\|^{1/2} \|\phi(y^* y) - \phi(y)^* \phi(y)\|^{1/2}$$

for any x and y . The following Kolmogorov-Sinai type result is a consequence of this inequality.

Lemma A.2. *If ω is a finite subset in A with the property that both x^* and $x^* x \in \omega$ whenever $x \in \omega$ and A is generated (as a C^* -algebra) by $\bigcup_{n \in \mathbb{Z}} \alpha^n(\omega)$, then we have $\text{ht}(\alpha) = \text{ht}(\alpha, \omega)$.*

Proof of Theorem. We deal with the following situation: $A_i \subset \mathbb{B}(\mathcal{H}_i)$ is a C^* -subalgebra and $\xi_i \in \mathcal{H}_i$ is a cyclic unit vector. We are given a finite selfadjoint subset ω_i of unitaries in A_i and ucp maps $\alpha_i: A_i \rightarrow \mathbb{M}_{r_i}$ and $\beta_i: \mathbb{M}_{r_i} \rightarrow \mathbb{B}(\mathcal{H}_i)$ such that $\|\beta_i \circ \alpha_i(a) - a\| < \varepsilon$ for $a \in \omega_i$. Fix $N \in \mathbb{N}$.

Since α_i extends to a ucp map from $\mathbb{B}(\mathcal{H}_i)$ and any ucp map from a von Neumann algebra into a matrix algebra can be approximated by normal ones (in the point-norm topology), we may assume that $\alpha_i = \alpha'_i \circ \Phi_i$, where $\Phi_i: \mathbb{B}(\mathcal{H}_i) \rightarrow \mathbb{B}(\mathcal{H}'_i)$ is the compression onto a finite dimensional subspace $\mathcal{H}'_i \subset \mathcal{H}_i$ and $\alpha'_i: \mathbb{B}(\mathcal{H}'_i) \rightarrow \mathbb{M}_{r_i}$ is a ucp map. Applying the Stinespring theorem to α'_i and β_i , we obtain an isometry $S_i: \ell_2^{r_i} \rightarrow \mathcal{H}_i \otimes H_i^S$ (with some Hilbert space H_i^S) such that $\alpha_i(x) = S_i^*(x \otimes 1_{H_i^S}) S_i$ for all $x \in \mathbb{B}(\mathcal{H}_i)$, and an isometry $T_i: \mathcal{H}_i \rightarrow \ell_2^{r_i} \otimes H_i^T$ (with some Hilbert space H_i^T) such that $\beta_i(x) = T_i^*(x \otimes 1_{H_i^T}) T_i$. Defining the isometry $V_i: \mathcal{H}_i \rightarrow \mathcal{H}_i \otimes \mathcal{K}_i$ with $\mathcal{K}_i = H_i^S \otimes H_i^T$ by $V_i = (S_i \otimes 1_{H_i^T}) \circ T_i$, we have $\|V_i^*(a \otimes 1_{\mathcal{K}_i}) V_i - a\| < \varepsilon$ for $a \in \omega_i$.

We define ‘length’ of vectors in \mathcal{H}_i as in the group case: define subspaces by $E_i^0 = \mathbb{C}\xi_i$, $E_i^k = \text{span } \omega_i E_i^{k-1}$ for $k = 1, \dots, N-1$ and $E_i^N = \mathcal{H}_i$, and put $F_i^k = E_i^k \ominus E_i^{k-1}$ for $k = 1, \dots, N$.

Let $(\mathcal{H}, \xi) = (\mathcal{H}_1, \xi_1) * (\mathcal{H}_2, \xi_2)$ and $\mathcal{K} = \mathbb{C}\xi_{\mathcal{K}} \oplus \mathcal{K}_1 \oplus \mathcal{K}_2$ and let $\chi_k = N^{-1/2} \sum_{p=1}^k \delta_p \in \ell_2(\mathbb{N})$. We define an isometry

$$V: \mathcal{H} \rightarrow \mathcal{H} \otimes \mathcal{K} \otimes \mathcal{H} \otimes \ell_2(\mathbb{N})$$

by

$$V\xi = \xi \otimes \xi_K \otimes \xi \otimes \chi_N,$$

and, for $\zeta_1 \otimes \cdots \otimes \zeta_l \in F_{i_1}^{k_1} \otimes \cdots \otimes F_{i_l}^{k_l}$,

$$\begin{aligned} V\zeta_1 \otimes \cdots \otimes \zeta_n &= V_{i_1}\zeta_1 \otimes (\zeta_2 \otimes \cdots \otimes \zeta_l) \otimes \chi_{k_1} \\ &\quad + (\zeta_1 \otimes V_{i_2}\zeta_2) \otimes (\zeta_3 \otimes \cdots \otimes \zeta_l) \otimes \chi_{k_2} \\ &\quad + \cdots + (\zeta_1 \otimes \cdots \otimes \zeta_{d-2} \otimes V_{i_{d-1}}\zeta_{d-1}) \otimes (\zeta_d \otimes \cdots \otimes \zeta_l) \otimes \chi_{k_{d-1}} \\ &\quad + (\zeta_1 \otimes \cdots \otimes \zeta_{d-1} \otimes V_{i_d}\zeta_d) \otimes (\zeta_{d+1} \otimes \cdots \otimes \zeta_l) \otimes \chi_{N-(k_1+\cdots+k_{d-1})} \end{aligned}$$

if $k_1 + \cdots + k_{d-1} < N \leq k_1 + \cdots + k_d$, (here $\zeta_{d+1} \otimes \cdots \otimes \zeta_l$ in the last term should be ξ when $d = l$) and

$$\begin{aligned} V\zeta_1 \otimes \cdots \otimes \zeta_n &= V_{i_1}\zeta_1 \otimes (\zeta_2 \otimes \cdots \otimes \zeta_l) \otimes \chi_{k_1} \\ &\quad + \cdots + (\zeta_1 \otimes \cdots \otimes \zeta_{l-1} \otimes V_{i_l}\zeta_l) \otimes \xi \otimes \chi_{k_l} \\ &\quad + (\zeta_1 \otimes \cdots \otimes \zeta_l) \otimes \xi_K \otimes \xi \otimes \chi_{N-(k_1+\cdots+k_l)} \end{aligned}$$


if $k_1 + \cdots + k_l < N$. We note that $\zeta_1 \otimes V_{i_2}\zeta_2 \in \mathcal{H}_{i_1}^\circ \otimes \mathcal{H}_{i_2} \otimes \mathcal{K}_{i_2}$, e.g., should be understood as an element in $\mathcal{H} \otimes \mathcal{K}$ by the usual convention that $\mathcal{H}_i^\circ \otimes \mathbb{C}\xi_j \ni \zeta \otimes \xi_j = \zeta \in \mathcal{H}_i^\circ \subset \mathcal{H}$.

Let $\Phi: \mathbb{B}(\mathcal{H}) \rightarrow \mathbb{B}(\mathcal{H})$ be the ucp map given by $\Phi(a) = V^*(a \otimes 1_{\mathcal{K} \otimes \mathcal{H} \otimes \ell_2(\mathbb{N})})V$. Then, a brute force computation shows that

$$\|\Phi(a) - a\| < 3(\varepsilon + \frac{1}{N})$$

for $a \in \bigcup_{i=1}^m \omega_i$. Here, we used the crucial fact that $aF_i^k \subset F_i^{k-1} \oplus F_i^k \oplus F_i^{k+1}$ for $a \in \omega_i$ and $k = 1, \dots, N$. Since $V_i^*(a \otimes 1_{\mathcal{K}_i})V_i = V_i^*(e_i \otimes 1_{H_i^T})(a \otimes 1_{H_i^S} \otimes 1_{H_i^T})(e_i \otimes 1_{H_i^T})V_i$ for the rank r_i projection e_i of $\mathcal{H}_i \otimes H_i^S$ onto $S_i\ell_2^{r_i}$, and $\dim E_i^{N-1} \leq (|\omega| + 1)^N$, we have the following estimate of the rank r of the ucp map Φ (i.e., Φ factors through M_r):

$$r \leq 2N(|\omega| + 1)^{N^2} \max\{r_1, r_2\}.$$

Combined with Lemma A.2, this proves Theorem A.1. 

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